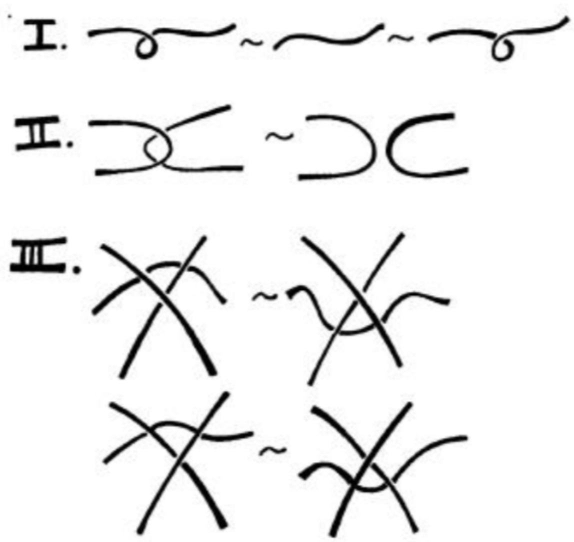
QUANDLES AND KNOT INVARIANTS I

LOS ANGELES MATH CIRCLE
ADVANCED 2
MAY 10, 2020

1. Introduction

A **knot** is a circle "embedded" in three-dimensional space. A **link** is a collection of several knots. We say two knots (or two links) are equivalent, denoted $\sim$, if we can smoothly deform one into the other.



We consider a **diagram** of a knot by projecting it onto a plane and indicating under/over at each crossing. Reidemeister proved that two knots are equivalent if and only if their corresponding diagrams can be transformed into each other using a sequence of the three **Reidemeister moves** shown above.

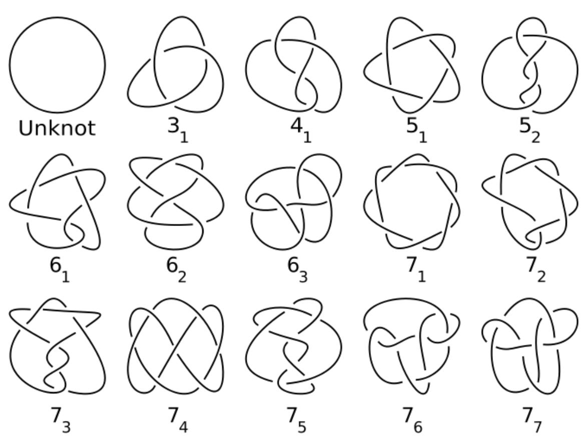
Problem 1. The standard circle circle is called the unknot. Is it true that the circle is equivalent to the following knots? If so, draw a sequence of Reidemeister moves transforming each to the unknot.

(a)

(b) 

Problem 2. Let E be the figure-eight knot and consider its mirror image E^* (reverse all the crossings). Draw a sequence of Reidemeister moves to show that $E \sim E^*$.

Problem 3. Let K be any knot and K^* its mirror image. Is it always true that would be $K \sim K^*$? Below are some famous examples of knots to help. Justify your answer.



Problem 4. (Challenge) Give a sequence of Reidemeister moves to show this knot is equivalent to the unknot.

We can give any link an **orientation** by assigning a direction to each link component. We draw this orientation with an arrow. We also record whether each crossing is "positive" or "negative:"

Problem 5. The **writhe** of an oriented link diagram is the number of positive crossings minus the number of negative crossings. Show that the writhe of a link diagram is invariant under Reidemeister moves II and III, but not I.

2. Quandles

Definition 1. A *quandle* is a set X with two binary operations, Q and N (which we pronounce quandle and unquandle), satisfying the following axioms:

(1) The quandle operation preserves identity: For any $x \in X$, $xQx = x$.
(2) The quandle and unquandle operations undo each other: For any $x, y \in X$, $(xQy)Nx = y = xQ(yNx)$.
(3) The quandle operations distribute over itself: For any $x, y, z \in X$, $xQ(yQz) = (xQy)Q(xQz)$ and $(xNy)Nz = (xNz)N(yNz)$.

A set with operations satisfying the latter two axioms is called a "Rack," while a set with an operation satisfying the last operation is called a "Shelf" - but these names are mostly historical oddities, as we'll only be considering quandles today.

Problem 6. Let X be any set and define $xQy = y$, and $xNy = x$ for any elements x, y of X. Verify Q and N satisfy the axioms. Note we have $(xQy)Nx = y$ and $xQ(yNx) = y$, yet we don't have $xN(yQx) = y$ nor do we have $(xNy)Qx = y$. (Only QN works!)

Problem 7. Prove that for any quandle (X, Q, N) and $y \in X$, $y \in X$, $yNy = y$. (Hint: Evaluate $(yQy)Ny$ in two ways.)

Problem 8. Let (X, Q, N) be a quandle. Show that if a and b are two distinct elements of X, aQb cannot be a. So, if X has more than one element, we cannot switch the roles of Q and N in the previous example (Problem 6).

Problem 9. (Challenge) Let $\mathrm{GL}_2(\mathbb{R})$ be the set of 2 by 2 invertible matrices with real entries. For matrices A, B, define $AQB = ABA^{-1}$ and $ANB = B^{-1}AB$. Show Q and N satisfy the above axioms. The two matrices get tangled in a more complicated way than Problem 6, and NA untangles AQ.

Problem 10. (a) Show that if $aQb = aQc$, then $b = c$. (Note that this is not true of $bQa = cQa$.)
(b) Show that if X is associative, then $aQb = b$ for all $a, b \in X$.

Problem 11. Prove that for any $a, b \in X$, there is a unique $x \in X$ such that $aQx = b$.

In light of Problem 11, we can replace the second quandle axiom with the uniqueness property. A quandle X is a set with operation Q such that:

(1) The quandle operation preserves identity: For any $x \in X$, $xQx = x$.
(2) For any $a, b \in X$, there is a unique $x \in X$ such that $aQx = b$.
(3) The quandle operation distributes over itself: For any $x, y, z \in X$, $xQ(yQz) = (xQy)Q(xQz)$.

Problem 12. Two quandles are considered the same if one can be obtained from the other by renaming the elements.

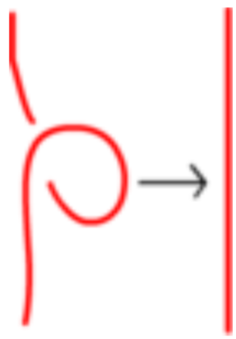
(a) Find all the different quandles of with one element.
(b) Find all the different quandles of with two elements.
(c) (Challenge) Find all the different quandles of with three elements.

Problem 13. Prove that the following are examples of quandles:

(a) The set of non-negative integers less than n, $\{0, 1, 2, \ldots, n-1\}$ with the operation $aQb = 2a - b \mod n$.
(b) The set of points in the plane, with aQb being the point on the opposite side of a from b, the same distance away.
(c) The set of points on the sphere, with aQb being the point that b is sent to when the sphere rotates $180°$ around a.

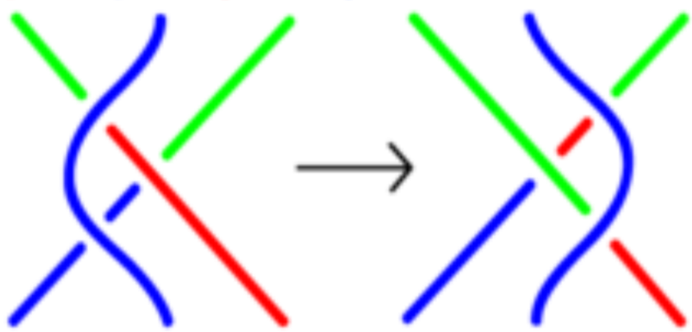
3. Knot Invariants

We have seen a bit of knot theory before, but here's a reminder: a knot is a continuous closed curve in 3 dimensional space that is not allowed to intersect itself. In order to draw knots in two dimensions, we project them onto the plane, with the requirement that intersections only have two arcs hitting at a point, and always indicating which is on top. While two different pictures might represent the same knot, by applying or reversing the Reidemeister moves:



Reidemeister Move 1

Reidemeister Move 2



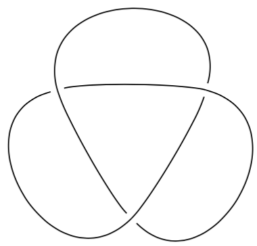
Reidemeister Move 3

we can get between any two depictions of the same knot.

Definition 2. A *knot invariant* is something (like a number, or perhaps something more complicated) associated to a knot diagram, that does not change when any of the Reidemeister moves are applied.

Definition 3. A knot diagram is said to be *tricolorable* if you can assign one of three colors to each unbroken curve in the diagram so that each crossing has either 1 or 3, but never 2, distinct colors, and not all curves are given the same color.

Problem 14. Which of these knots are tricolorable?



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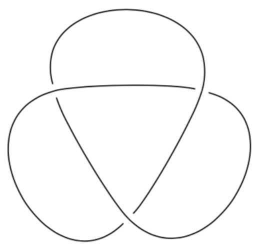
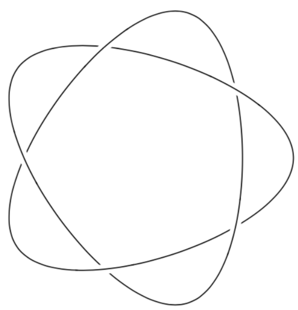
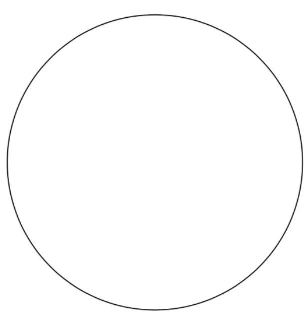






Problem 15. Prove that tricolorability is an invariant — check that if a knot is tricolored, and you apply or reverse any of the Reidemeister moves, it can remain tricolored.

4. The Knot Quandle

Given any knot, it turns out that you can build a quandle!

Definition 4. The *knot quandle* of a given knot diagram is constructed by labeling each of the arcs in the knot diagram, making sure to label on the same side of the curve as you follow it around. These labels generate the quandle; it is subject to relations obtained at each of the crossings. Assuming the top strand is labeled b and that label is on the righthand side of the strand (rotate the picture if necessary), and that the label on the left strand is a and the right strand is c: if the labels a and c are below their strands, we get the relation $bQa = c$, while if the labels a and c are above their strands, we get the relation $a = cNb$.

Problem 16. Draw what the above relations mean at a crossing.

Problem 17. Determine the knot quandle for each of the knots from Problem 5.

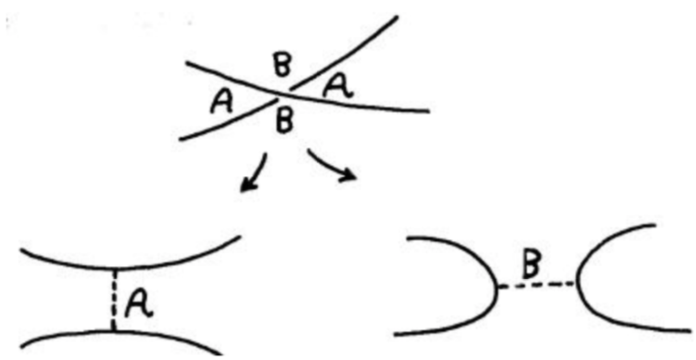
Problem 18. Prove that the knot quandle is a knot invariant.

QUANDLES AND KNOT INVARIANTS II

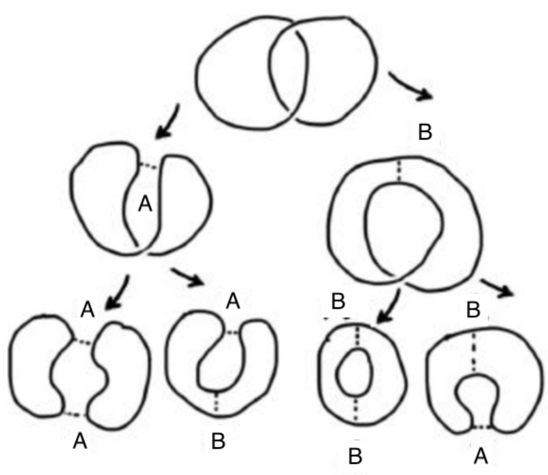
LOS ANGELES MATH CIRCLE
ADVANCED 2
MAY 17, 2020

1. Kauffman Bracket

In general, it is difficult to figure out when two knots are not equivalent. The earliest examples of knot invariants, like tricolorability, are not complex enough for many examples. Our goal in this section is to construct a polynomial for each knot diagram that is invariant under Reidemeister moves. We start by cutting our knot (or link) K into simple pieces. At each crossing in our knot diagram, we make two new descendant links by performing a type-A or type-B splicing:



We keep track of each splicing and its type by labeling it A or B. Continue this way until we reach the **primitive descendants**, the descendants with no crossings. For example, we calculate the four primitive descendants of the "Hopf link":



In the following, we will consider A and B as commuting variables. Define the **Kauffman bracket** $\langle K\rangle$ as follows:

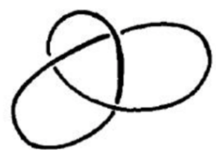
(1) If P is a primitive descendant of K, define the bracket $\langle K|P\rangle$ as the product of all labels of P. For example, the leftmost primitive descendant of the Hopf link has bracket $\langle K|P\rangle = B^2$.
(2) Let $||P||$ be the number of components (disjoint knots) in P, minus 1.

(3) Define
$$\langle K \rangle = \sum_P \langle K|P \rangle d^{||P||}$$
where the sum is over all primitive descendants P and d is another variable.

Problem 1. Find the Kauffman bracket of the Hopf link given the decomposition above.

Problem 2. Let K be a knot. What is $\langle \circ K \rangle$, where $\circ K$ is the knot formed by adding a disjoint unknot to K?

Problem 3. Let T be the trefoil knot shown below. Compute $\langle T \rangle$.

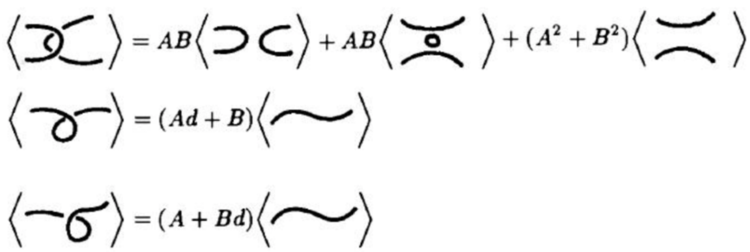


Problem 4. Use the definition to prove that

The notation means that the diagram in the bracket is a part of a larger diagram, and we only change the part shown in the bracket.

Problem 5. (Riedemeister II Invariance).

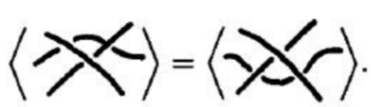
(a) Prove the following identities:



(b) Declare $B = A^{-1}$ and $d = -A^2 - A^{-2}$. With these equalities in place, show that the bracket becomes invariant under Reidemeister II moves.

From now on, we assume $B = A^{-1}$ and $d = -A^2 - A^{-2}$.

Problem 6. (Reidemeister III Invariance). Show that the bracket is invariant under Reidemeister III moves, i.e.



Recall, the **writhe** of an oriented link diagram is the number of positive crossings minus the number of negative crossings.

To make the bracket invariant under Reidemeister I, we need to modify it further. Let K be an oriented link and define the **normalized bracket** of

$$\mathcal{L}_K(A) = (-A^3)^{-w(K)}\langle K\rangle$$

where $w(K)$ is the writhe of K.

Problem 7. Prove that the normalized bracket is invariant under Reidemeister I moves. Conclude that the normalized bracket is an invariant under all Reidemeister moves and hence determines an oriented knot invariant!

Problem 8. (a) Determine the relationship between $\mathcal{L}_K(A)$ and $\mathcal{L}_{K^*}(A)$.
(b) Compute $\langle T^*\rangle$ where T^* is the mirror image of the trefoil.
(c) Compare with $\langle T\rangle$ to conclude that $T \not\sim T^*$.

Problem 9. Consider the collection of knots K_n:

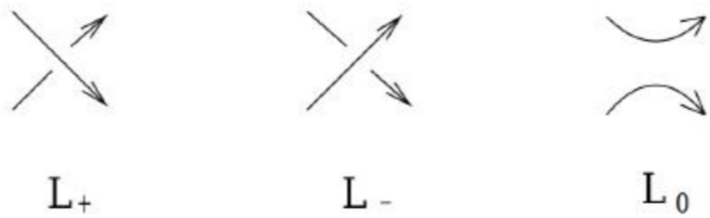
K_1 K_2 K_3 K_4 ...

(a) Find a recursive formula for $\langle K_n\rangle$.
(b) Show that K_n is not equivalent to K_n^* for any $n > 1$.

2. Jones Polynomial

The Jones polynomial is another (oriented) knot invariant $V_K(t)$ discovered by Vaughan Jones in quite a different context. Its axioms are:

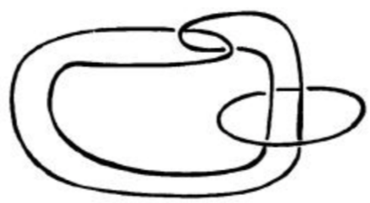
(1) $V_\circ(t) = 1$
(2) If $K \sim K'$ then $V_K(t) = V_{K'}(t)$.
(3) $t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t)$ where L_+, L_-, L_0 are as below:



The existence of V_K is not obvious; the following problem relates it to our earlier construction.

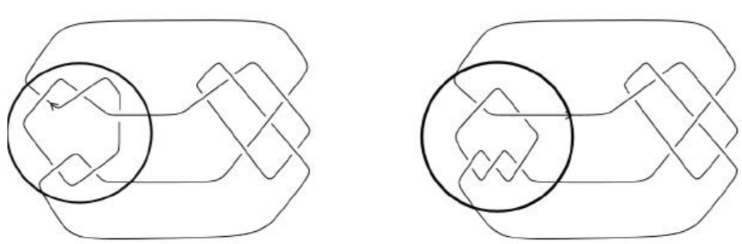
Problem 10. Show that $\mathcal{L}_K(t^{-1/4})$ satisfies (1)-(3). As a result, we can take $V_K(t) = \mathcal{L}_K(t^{-1/4})$.

Problem 11. Let W be the Whitehead link shown below. Prove that $W \not\sim \circ\circ$ by choosing an orientation and computing the Jones polynomial.



Problem 12. Prove using either the Jones polynomial or the Kauffman bracket that the following knot is not equivalent to the unknot.

Problem 13. (Challenge). Unfortunately, Jones' polynomial cannot always detect nonequivalent knots. It turns out knots (a) and (b) are non-equivalent. The transformation shown in the circled region is known as a Conway mutation. Prove that the Conway mutation does not change the Jones polynomial of knots (a) and (b).



(a) Kinoshita-Terasaka knot (b) Conway knot

Problem 14. (Challenge). Choose an orientation of the following two knots and compute their Jones polynomials. Are they equivalent knots? (For the answer see knotplot.com/perko/.)